Compressed Sensing and Generative Models

Ashish Bora    Ajil Jalal    Eric Price    Alex Dimakis

UT Austin
Talk Outline

1. Compressed sensing

2. Using generative models for compressed sensing

3. Learning generative models from noisy data
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1. Compressed sensing

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3. Learning generative models from noisy data
Compressed Sensing

- Want to recover a signal (e.g., an image) from noisy measurements.

Linear measurements: see $y = Ax$, for $A \in \mathbb{R}^{m \times n}$.

How many measurements to learn the signal?
Compressed Sensing

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Examples:

- Medical Imaging
- Astronomy
- Single-Pixel Camera
- Oil Exploration
- Genetic Testing
- Streaming Algorithms
Compressed Sensing

- Want to recover a signal (e.g., an image) from noisy measurements.

Medical Imaging  Astronomy  Single-Pixel Camera  Oil Exploration

Genetic Testing  Streaming Algorithms

- *Linear* measurements: see $y = Ax$, for $A \in \mathbb{R}^{m \times n}$.
- How many measurements $m$ to learn the signal?
Compressed Sensing

- Given linear measurements $y = Ax$, for $A \in \mathbb{R}^{m \times n}$.
- How many measurements $m$ to learn the signal $x$?

- Naively: $m \geq n$ or else underdetermined: multiple $x$ possible.

- But most $x$ aren’t plausible.

- This is why compression is possible.

Ideal answer: $m \approx (\text{information in image})/(\text{new info. per measurement})$.
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![Image 1]
![Image 2]
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- Ideal answer:

$$m \approx \frac{\text{(information in image)}}{\text{(new info. per measurement)}}$$
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- Image “compressible” $\implies$ information in image is small.
Compressed Sensing

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- Image “compressible” $\implies$ information in image is small.
- Measurements “incoherent” $\implies$ most info new.
Compressed Sensing

- Want to estimate $x \in \mathbb{R}^n$ from $m \ll n$ linear measurements.
Compressed Sensing

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Suggestion: the “most compressible” image that fits measurements.
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How should we formalize that an image is “compressible”? 

Short JPEG compression

Intractible to compute.

Standard compressed sensing: sparsity
Compressed Sensing

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- Short JPEG compression
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- Standard compressed sensing: *sparsity* in some basis
Approximate sparsity is common

Coefficient decay (log log plot)

Renormalized magnitude of $i$th largest coordinate

Coefficient decay (log log plot)

Music frequencies

Wikipedia inlinks

Image wavelet

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Sample complexity of sparse recovery

\[ m \approx \frac{\text{(information in image)}}{\text{(new info. per measurement)}} \]
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- If 99\% of energy in largest \( k \) coordinates…
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- If 99% of energy in largest \( k \) coordinates...
- Information in image is \( \approx \log \left( \frac{n}{k} \right) \approx k \log n \)
Sample complexity of sparse recovery

\[ m \approx \frac{\text{(information in image)}}{\text{(new info. per measurement)}} \]

- If 99% of energy in largest \( k \) coordinates...
- Information in image is \( \approx \log \left( \binom{n}{k} \right) \approx k \log n \)
- New info. per measurement is hopefully \( \approx \log 100 = \Theta(1) \)
Compressed Sensing Formalism

“Compressible” = “sparse”

- Want to estimate $x$ from $y = Ax + \eta$, for $A \in \mathbb{R}^{m \times n}$.

- For this talk: ignore $\eta$, so $y = Ax$.

- Goal: $\hat{x}$ with $\|x - \hat{x}\|_2 \leq O(1) \cdot \min_{k}$-sparse $x'$ $\|x - x'\|_2$ with high probability.

- Reconstruction accuracy proportional to model accuracy.

- Theorem [Cand` es-Romberg-Tao 2006]: $m = \Theta(k \log(n/k))$ suffices for (1).

- Such an $\hat{x}$ can be found efficiently with, e.g., the LASSO.
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**Theorem [Candès-Romberg-Tao 2006]**

- $m = \Theta(k \log(n/k))$ suffices for (1).
- Such an $\hat{x}$ can be found efficiently with, e.g., the LASSO.
Lower bound: \( k = 1 \)

- Hard case: \( x \) is random \( e_i \) plus Gaussian noise \( w \) with \( \|w\|_2 \approx 1 \).

- Robust recovery must locate \( i \).

- Observations \( \langle v, x \rangle = v_i + \langle v, w \rangle = v_i + \frac{\|v\|_2}{\sqrt{n}} z, \) for \( z \sim \mathcal{N}(0, 1) \).
1-sparse lower bound

P-Woodruff '11

- Observe $\langle v, x \rangle = v_i + \frac{\|v\|_2}{\sqrt{n}} z$, where $z \sim N(0, \Theta(1))$
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Shannon 1948: AWGN channel capacity is

$$I(i, \langle v, x \rangle) \leq \frac{1}{2} \log(1 + \text{SNR})$$

where $\text{SNR}$ denotes the “signal-to-noise ratio,”
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- (info. per measurement) $= O(1)$
Lower bound
P-Woodruff '11

\[ m \approx \frac{\text{(information in image)}}{\text{(new info. per measurement)}} \]

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Lower bound

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m \approx \frac{(\text{information in image})}{(\text{new info. per measurement})}
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- (info. per measurement) = \(O(1)\)
- \(k = 1: (\text{information in image}) = \log n \implies m = \Omega(\log n)\)

\[
\Rightarrow m = \Omega(\log n)
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Lower bound
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- (info. per measurement) = \(O(1)\)
- \(k = 1\): (information in image) = \(\log n\) \(\Rightarrow\) \(m = \Omega(\log n)\)

General \(k\): \(m = \Omega(\log \binom{n}{k}) = \Omega(k \log(n/k))\).
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Alternatives to sparsity?

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- Best way to model images in 2019?

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- Best way to model images in 2019?
  - In particular: \textit{generative models}. 

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Random noise $z$
Generative Models

Random noise $z$
Generative Models

Random noise $z$ → Image
Training Generative Models

Random noise $z$ → Figure
Training Generative Models

Random noise $z$ → Image

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Training Generative Models

Random noise $z$ → → → Image
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Random noise $z$
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Random noise $z$ \rightarrow \rightarrow \rightarrow n \rightarrow

Image

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Training Generative Models

Random noise $z$ $\rightarrow$ $k$ $\rightarrow$ $n$ $\rightarrow$ Image

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Generative Models

- Want to model a distribution $\mathcal{D}$ of images.
Generative Models

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- Function $G : \mathbb{R}^k \rightarrow \mathbb{R}^n$. 

Generative Adversarial Networks (GANs) [Goodfellow et al. 2014]:
- Karras et al., 2018
- Faces
- Schawinski et al., 2017
- Astronomy
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- Particle Physics

Variational Auto-Encoders (VAEs) [Kingma & Welling 2013].
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- Want to model a distribution $D$ of images.
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Suggestion for compressed sensing

Replace “$x$ is $k$-sparse” by “$x$ is in range of $G : \mathbb{R}^k \rightarrow \mathbb{R}^n$.”

- Variational Auto-Encoders (VAEs) [Kingma & Welling 2013].
Our Results

“Compressible” = “near range(G)”

- Want to estimate $x$ from $y = Ax$, for $A \in \mathbb{R}^{m \times n}$.

Main Theorem I: $m = O(kd \log n)$ suffices for (2).

$G$ is a $d$-layer ReLU-based neural network.

When $A$ is random Gaussian matrix.

Main Theorem II: For any Lipschitz $G$, $m = O(k \log r \delta)$ suffices.

Morally the same $O(kd \log n)$ bound.
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- Goal: $\hat{x}$ with

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\|x - \hat{x}\|_2 \leq O(1) \cdot \min_{k\text{-sparse } x'} \|x - x'\|_2
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(2)
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“Compressible” = “near range(\(G\))”

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\|x - \hat{x}\|_2 \leq O(1) \cdot \min_{x' = G(z'), \|z'\|_2 \leq r} \|x - x'\|_2 + \delta
\]  

(2)

- Main Theorem I: $m = O(kd \log n)$ suffices for (2).
  - $G$ is a $d$-layer ReLU-based neural network.
  - When $A$ is random Gaussian matrix.

- Main Theorem II:
  - For any Lipschitz $G$, $m = O(k \log \frac{rL}{\delta})$ suffices.
Our Results

“Compressible” = “near range($G$)"

- Want to estimate $x$ from $y = Ax$, for $A \in \mathbb{R}^{m \times n}$.
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$$\|x - \hat{x}\|_2 \leq O(1) \cdot \min_{x' = G(z'), \|z'\|_2 \leq r} \|x - x'\|_2 + \delta$$  \hspace{1cm} (2)

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$$

$m = O(kd \log n)$ suffices for $d$-layer $G$. 

Compared to $O(k \log n)$ for sparsity-based methods. $k$ here can be much smaller.
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"Compressible" = "near range(\(G\))"

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  - Just like for training, no proof this converges
  - Approximate solution approximately gives (3)
  - Can check that \(\|\hat{x} - x\|_2\) is small.
  - In practice, optimization error seems negligible.
Related Work

- Projections on manifolds (Baraniuk-Wakin ’09, Eftekhari-Wakin ’15)
Related Work

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  - Conditions on manifold for which recovery is possible.
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  - Conditions on manifold for which recovery is possible.
- Deep network models (Mousavi-Dasarathy-Baraniuk ’17, Chang et al ’17)
  - Train deep network to encode and/or decode.
Experimental Results

Faces: $n = 64 \times 64 \times 3 = 12288$, $m = 500$

Original

![Image of original faces]
Experimental Results

Faces: \( n = 64 \times 64 \times 3 = 12288 \), \( m = 500 \)
Experimental Results

Faces: $n = 64 \times 64 \times 3 = 12288, \ m = 500$

- Original
- Lasso (DCT)
- Lasso (Wavelet)
- DCGAN

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Experimental Results

MNIST: $n = 28 \times 28 = 784$, $m = 100$. 

Original

Lasso

VAE
Experimental Results

**MNIST**

![MNIST Reconstruction Error Plot](image)

**Faces**

![Faces Reconstruction Error Plot](image)
Proof Outline (ReLU-based networks)

- Show range($G$) lies within union of $n^{dk}$ $k$-dimensional hyperplane.
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Layer 1’s output lies within a union of $n^k$ $k$-dimensional hyperplanes.
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Layer 1’s output lies within a union of \( n^k \) \( k \)-dimensional hyperplanes.

- Induction: final output lies within \( n^{dk} \) \( k \)-dimensional hyperplanes.
Proof of Lemma

Layer 1’s output lies within a union of $n^k$ $k$-dimensional hyperplanes.

- $z$ is $k$-dimensional.
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- How many different patterns can sign($A_1z$) take?
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\]

- \( n \) half-spaces divide \( \mathbb{R}^k \) into less than \( n^k \) regions.

Therefore \( d \)-layer network has \( n^{dk} \) regions.
Summary of Compressed Sensing with Generative Models

\[ m \approx \frac{\text{(information in image)}}{\text{(new info. per measurement)}} \]
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Follow-up work

Theorem

For any $L$-Lipschitz $G : \mathbb{R}^k \to \mathbb{R}^n$, recovering $\hat{x}$ from $Ax$ satisfying

$$\|x - \hat{x}\|_2 \leq O(1) \cdot \min_{x' = G(z'), \|z'\|_2 \leq r} \|x - x'\|_2 + \delta$$

requires $m = O(k \log \frac{rL}{\delta})$ linear measurements.
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- Matching lower bounds: [Liu-Scarlett] (Poster outside!) and [Kamath-Karmalkar-P]
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- Better results with better \( G \): (Asim-Ahmed-Hand ’19)
- Provably fast for random networks (Hand-Voroninski ’18)
Extensions

- Inpainting:
Extensions

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Extensions

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  - $A$ is diagonal, zeros and ones.
Extensions

- Inpainting:
  - A is diagonal, zeros and ones.

- Deblurring:
Extensions

- **Inpainting:**
  
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  ![Inpainting Example](image1)
  ![Deblurring Example](image2)
Extensions

- Inpainting:
  - A is diagonal, zeros and ones.

- Deblurring:

- Can apply even to nonlinear—but differentiable—measurements.
Talk Outline

1. Compressed sensing

2. Using generative models for compressed sensing

3. Learning generative models from noisy data
Where does the generative model come from?
Where does the generative model come from?

Training from lots of data.
Where does the generative model come from?

Training from lots of data.

Problem
If measuring images is hard/noisy, how do you collect a good data set?
Where does the generative model come from?

Training from lots of data.

Problem

If measuring images is hard/noisy, how do you collect a good data set?

Question

Can we learn a GAN from incomplete, noisy measurements?
GAN Architecture

$Z$
GAN Architecture

\[ Z \rightarrow G \]
GAN Architecture

\[ \text{Z} \rightarrow G \rightarrow \text{Generated image} \]
GAN Architecture

$Z \rightarrow G \rightarrow \text{Generated image}$

$\text{Real image}$
GAN Architecture

Z \rightarrow G \rightarrow \text{Generated image}

\text{Real image} \rightarrow D

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GAN Architecture

Z → G → Generated image

G → Real image

D → Real?

Ashish Bora, Ajil Jalal, **Eric Price**, Alex Dimakis (UT Austin)
Generator $G$ wants to fool the discriminator $D$. 
Generator $G$ wants to fool the discriminator $D$.
If $G, D$ infinitely powerful: only pure Nash equilibrium when $G(Z)$ equals true distribution.
GAN Architecture

- Generator $G$ wants to fool the discriminator $D$.
- If $G$, $D$ infinitely powerful: only pure Nash equilibrium when $G(Z)$ equals true distribution.
- Empirically works for $G$, $D$ being convolutional neural nets.
GAN training

Discriminator must distinguish real measurements from simulated measurements of fake images. Can try this for any measurement process you understand.

Compatible with any GAN generator architecture.

Ashish Bora, Ajil Jalal, Eric Price, Alex Dimakis (UT Austin)

Compressed Sensing and Generative Models
GAN training

- $Z$ is input to the generator $G$.
- $G$ generates an image.
- The discriminator $D$ is trained to distinguish between real and simulated measurements.
- The discriminator takes both real and simulated images as input and outputs whether the image is real or not.

This process can be applied to any measurement process that you understand.

Compatible with any GAN generator architecture.
AmbientGAN training

The AmbientGAN training process involves a generator (G) that takes a random noise vector (Z) as input and produces a generated image. A discriminator (D) is used to distinguish between real measurements and simulated measurements of fake images. The discriminator must be able to distinguish real measurements from simulated measurements of fake images. This approach is compatible with any GAN generator architecture.

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AmbientGAN training

- Generated image: $G$(Z)
- Simulated measurement: $f$
- Real measurement
- Discriminator $D$ must distinguish real measurements from simulated measurements of fake images.

Can try this for any measurement process $f$ you understand.

Compatible with any GAN generator architecture.

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Ambient GAN training

- Discriminator must distinguish real measurements from simulated measurements of fake images.

Can try this for any measurement process you understand.

Compatible with any GAN generator architecture.

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AmbientGAN training

- Discriminator must distinguish *real measurements* from *simulated measurements of fake images*
- Can try this for any measurement process $f$ you understand.
AmbientGAN training

- Discriminator must distinguish real measurements from simulated measurements of fake images.
- Can try this for any measurement process $f$ you understand.
- Compatible with any GAN generator architecture.
Measurement: Gaussian blur + Gaussian noise

- Gaussian blur + additive Gaussian noise attenuates high-frequency components.
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Wiener baseline: deconvolve before learning GAN.
Measurement: Gaussian blur + Gaussian noise

- Gaussian blur + additive Gaussian noise attenuates high-frequency components.
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- AmbientGAN better preserves high-frequency components.
Measurement: Gaussian blur $+$ Gaussian noise

- Gaussian blur $+$ additive Gaussian noise attenuates high-frequency components.
- Wiener baseline: deconvolve before learning GAN.
- AmbientGAN better preserves high-frequency components.
- Theorem: in the limit of dataset size and $G$, $D$ capacity $\to \infty$, Nash equilibrium of AmbientGAN is the true distribution.
Measurement: Obscured Square

- Obscure a random square containing 25% of the image.
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Inpainting followed by GAN training reproduces inpainting artifacts.
Obscure a random square containing 25% of the image.

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AmbientGAN gives much smaller artifacts.
Measurement: Obscured Square

- Obscure a random square containing 25% of the image.
- Inpainting followed by GAN training reproduces inpainting artifacts.
- AmbientGAN gives much smaller artifacts.
- No theorem: doesn’t know that eyes should have the same color.
Measurement: Limited View

- Motivation: learn the distribution of *panoramas* from the distribution of *photos*?
Measurement: Limited View

- Motivation: learn the distribution of *panoramas* from the distribution of *photos*?

  Measured

- Reveal a random square containing 25% of the image.
Measurement: Limited View

- Motivation: learn the distribution of *panoramas* from the distribution of *photos*?

**Measured**

![Image of a measured face]

**AmbientGAN**

![Image of AmbientGAN recovering a face]

- Reveal a random square containing 25% of the image.
- AmbientGAN still recovers faces.
Measurement: Dropout

- Drop each pixel independently with probability $p = 95\%$. 

Measured
Measurement: Dropout

- Drop each pixel independently with probability $p = 95\%$.
- Simple baseline does terribly.
Measurement: Dropout

- Drop each pixel independently with probability $p = 95\%$.
- Simple baseline does terribly.
- AmbientGAN can still learn faces.
Drop each pixel independently with probability $p = 95\%$.

Simple baseline does terribly.

AmbientGAN can still learn faces.

Theorem: in the limit of dataset size and $G, D$ capacity $\to \infty$, Nash equilibrium of AmbientGAN is the true distribution.
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Thank You